A lot of fudge around $A + B = C$

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version 1.0, January 7, 2023

Abstract

This paper reports on experiments searching for elliptic curves over $\mathbb{Q}$ with large Tamagawa products. The main idea is to look among curves related to good *abc*-triples.

1 Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$, with conductor $N = N(E)$ and Tamagawa product (also called 'fudge factor') $\tau = \tau(E)$. In my 1998 paper [dW2] I proved a (conditional) upper bound for τ in terms of N, namely

Lemma 1 (de Weger, 1998) *Szpiro's Conjecture implies that for all elliptic curves*

$$\tau \ll N^{\epsilon}.$$

This is short for: For each $\epsilon > 0$ there exists an absolute constant C_ϵ such that for all elliptic curves over $\mathbb{Q}$ it is true that $\tau < C_\epsilon N^\epsilon$. In fact, I proved a slightly better result, namely the existence (assuming Szpiro's Conjecture) of an absolute constant C such that $\tau < N^{C/\log\log N}$, but this was not needed in that paper. Actually I wanted τ to be small, as the goal was to find curves with exceptionally big Tate-Shafarevich groups, and the Birch and Swinnerton-Dyer Conjecture suggests that then τ being small is helpful.

Recently this little auxiliary lemma has received interest from, a.o., Hector Pasten, in relation to the *abc* Conjecture, see [P1], [P3]. In his paper [P2] Pasten improved upon Lemma 1 by the following conjecture.

Conjecture 2 (Pasten, 2020) *For all elliptic curves*

$$\tau \ll N^{\left(\frac{7}{3}\log 3+\epsilon\right)/\log\log N}.$$

Note that $\frac{7}{3}\log 3 = 2.563\ldots$, and note that I left out the term $+O_\epsilon(1)$ from [P2, Conjecture 1.1], as it seems superfluous. Pasten remarks that the constant $\frac{7}{3}\log 3$ might not be optimal, but also believes there is 'some evidence' for it. Anyway, large τ's are rare, as it is known [GOT] that the average τ is $1.8193\ldots$.

This leads me to define the *Tamagawa quality* of an elliptic curve by

$$q_\tau = \frac{\log\tau\log\log N}{\log N}.$$

It is the purpose of this note to report on first experimental results searching for elliptic curves with an exceptionally high Tamagawa quality q_τ, and also for elliptic curves with an exceptionally large Tamagawa product τ itself. The methods used here are restricted to picking the low hanging fruit only, and it is my hope that this note spurs interest from others to find better methods and results. Tables with all found curves can be found online [dW3].

2 Curves from the LMFDB and Cremona Databases

LMFDB [LMFDB] is a database with web interface containing a wealth of data on a.o. elliptic curves. In particular it contains a collection, called `ec_mwbsd`[1], of data related to the Birch and Swinnerton-Dyer Conjecture for 3824372 curves, containing for each curve a.o. the conductor and the Tamagawa product. So this is a good place to start. However, the web interface is not well suited for directly checking the Tamagawa quality for all curves in the database. I found it easiest to download the underlying database in two parts.

The full data for all 3064705 curves with conductor up to 500000 can be downloaded directly from John Cremona's database [C]. I did so, and found the following results (see the green dots in Figure 1 below).

- 10795 curves have $q_\tau > 1.5$,
- 135 curves have $q_\tau > 2$,
- the curve with largest $q_\tau = 2.30681$ is $y^2 + xy = x^3 - 1054050116x - 12046088636400$, with $N = 39270 = 2\,3\,5\,7\,11\,17$ and $\tau = 31104 = 2^7\,3^5$,
- the curve with largest $\tau = 87040 = 2^{10}\,5\,17$ is $y^2 + xy = x^3 - 4456595642213x - 15384863559508100000$, with $N = 364650 = 2\,3\,5^2\,11\,13\,17$ and $q_\tau = 2.26473$.
- Full tables `output_cremona_qua.txt`, `output_cremona_tam.txt` are on [dW3].

Note that Cremona's Database contains all curves with conductor up to 500000.

This leaves 759667 curves from the LMFDB `ec_mwbsd` collection with conductor between 500000 and 300000000. The web interface does allow an easy download of data through `https://www.lmfdb.org/EllipticCurve/Q/?conductor=500000-`, but this does not include Tamagawa products, one only gets conductors and Weierstrass coefficients a_1, a_2, a_3, a_4, a_6. However, computing τ by SageMath [S] then is trivial, with the code snippet

```
tau = EllipticCurve([a1,a2,a3,a4,a6]).tamagawa_product()
```

Doing this I found the following results.

- no curve has $q_\tau > 1.5$,
- the largest q_τ found is 1.22859,
- the largest τ found is 576.

This somewhat disappointing result is probably due to the fact that of the curves of conductor between 500000 and 300000000 only those of prime or 7-smooth conductor have been incorporated, whereas τ seems to get large only when the conductor has many small prime factors.

[1] `https://www.lmfdb.org/api/ec_mwbsd/`

3 Curves from *abc*-triples

3.1 *abc*-triples and Frey-Hellegouarch curves

Let a, b, c be a triple of positive integers satisfying $a+b=c$, $a<b$, and $\gcd(a,b)=1$. Its *radical* $r(a,b,c)$ is the product of the distinct prime divisors of a, b and c, i.e. $r(a,b,c) = \prod_{\text{prime } p|abc} p$.
Such a triple a, b, c is called an *abc-triple* if it satisfies $c > r(a,b,c)$. The *abc* Conjecture states that $c \ll r(a,b,c)^{1+\epsilon}$, and this has led to the definition of the *quality* of an *abc*-triple as $q(a,b,c) = \dfrac{\log c}{\log r(a,b,c)}$ (so a triple is an *abc*-triple if and only if $q(a,b,c) > 1$). This concept of quality, although not yet under that name, seems to appear for the first time in my paper [dW1]. Later also the *merit* $m(a,b,c) = (q(a,b,c)-1)^2 \log r(a,b,c) \log\log r(a,b,c)$ was introduced as an interesting measure for *abc*-triples. See Bart de Smit's website [dS] for a wealth of experimental information on high quality and high merit *abc*-triples. In this note I adopt the terminology *medium quality* for an *abc*-triple with $1.3 < q(a,b,c) < 1.4$, next to *high quality* for an *abc*-triple with $q(a,b,c) > 1.4$ (as used in [dW1]), and following [dS] I also use *high merit* for an *abc*-triple with $m(a,b,c) > 24$, and *unbeaten* if there is no *abc*-triple known with larger c and larger $q(a,b,c)$.

For an *abc*-triple a, b, c it makes sense to look at its Frey-Hellegouarch curve, defined by $y^2 = x(x-a)(x+b)$, because its conductor equals $r(a,b,c)$ up to a bounded power of 2, so that such elliptic curves have exceptionally small conductors precisely for good quality *abc*-triples.

The Birch and Swinnerton-Dyer Conjecture for elliptic curves over $\mathbb{Q}$ states

$$\Omega\tau|\text{Ш}| = \frac{T^2}{R} \lim_{s\to 1} \frac{L(s)}{(s-1)^r},$$

where $\Omega = \omega$ or 2ω for the period ω, Ш is the Tate-Shafarevich group, T is the order of the torsion group, R is the regulator, $L(s)$ is the L-series, and r is the rank of the elliptic curve. This conjecture is believed to hold for all elliptic curves, but in this note I restrict to Frey-Hellegouarch curves.

The conductor is not explicitly there in the Birch and Swinnerton-Dyer formula, but its influence comes via Ω, as it is known that $\Omega \ll \dfrac{\log c}{\sqrt{c}}$, which in the case of a good *abc*-triple implies $\Omega \ll N^{-1/2+\epsilon}$. In other words, the Frey-Hellegouarch curve then has an exceptionally small period, and the Birch and Swinnerton-Dyer Conjecture then suggests that this must be compensated for somewhere. The somewhat unusual way I have used above to present the Birch and Swinnerton-Dyer formula is suggestive for where I will be looking for this compensation.

The main idea of [dW2] was that if one can show that, next to Ω, also the Tamagawa product τ is small compared to the conductor, then one may expect large Ш's, and this idea turned out to be fruitful, see also [N], [DW], [B]. In this note I now complement this with the idea that, although there is a subpolynomial upper bound $\tau \ll N^{C/\log\log N}$, it may still occur that large τ accounts for a substantial part of this compensation of very small periods. In other words, one may expect big Tamagawa products also at Frey-Hellegouarch curves, and, like in [dW2], their quadratic twists and isogenous curves.

So this sets the program for the remainder of this note: to search for elliptic curves with large Tamagawa products τ, and large Tamagawa qualities q_τ, by looking at curves isogenous to quadratic twists of Frey-Hellegouarch curves for known good *abc*-triples. For those *abc*-triples the website of Bart de Smit [dS] is an amazingly good source.

Let's start with a picture giving an overview of the results.

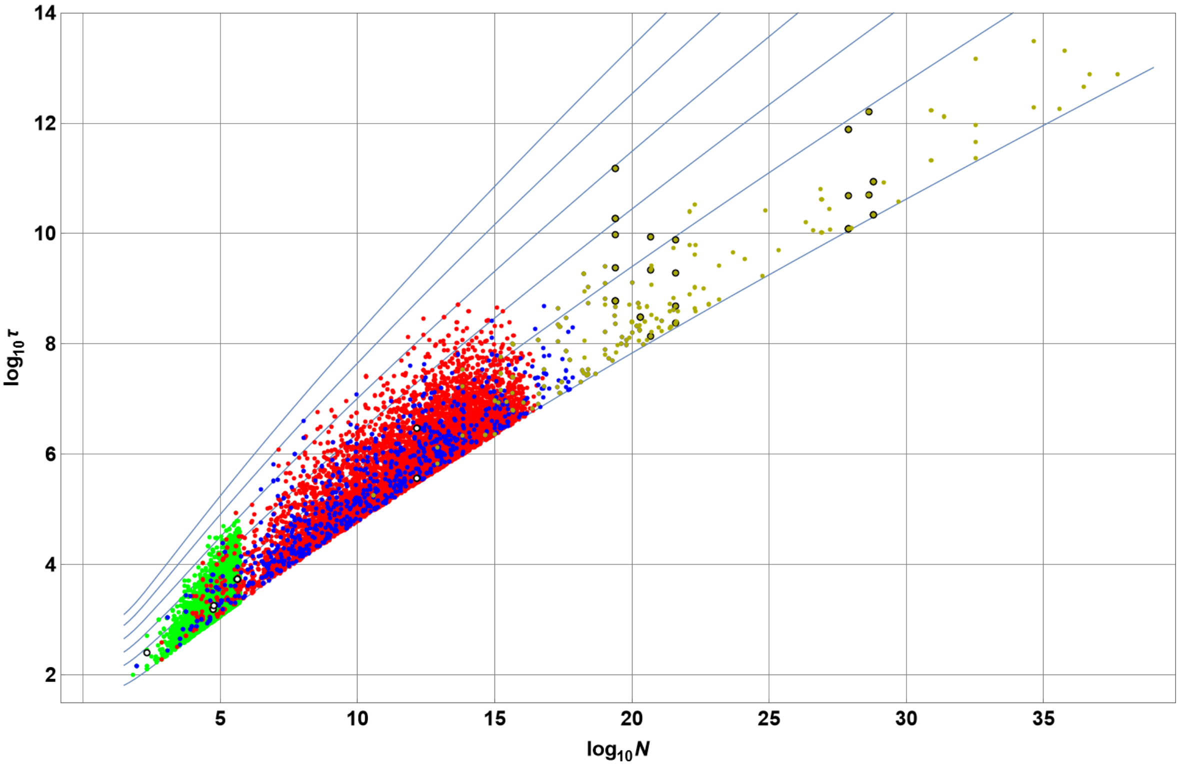

Figure 1: Elliptic curves with large Tamagawa product τ and Tamagawa quality $q_\tau > 1.5$.

Legend: curved lines: $q_\tau = \frac{7}{3}\log 3 = 2.563\ldots, 2.4, 2.2, 2, 1.8, 1.5,$
green dots: curves from the Cremona and LMFDB databases,
yellow dots: curves from high merit *abc*-triples,
red dots: curves from medium quality *abc*-triples,
blue dots: curves from high quality *abc*-triples,
black circles: curves from 'triples from triples'.

3.2 Curves from high quality *abc*-triples

There are 241 high quality *abc*-triples known, nicely presented as such on [dS]. For each I took the twisted Frey-Hellegouarch curves $dy^2 = x(x-a)(x+b)$ for $d = \pm1, \pm2, \pm3, \pm5, \pm6$, and some isogenous curves, computed by SageMath [S] as follows:

```
E = EllipticCurve([0, d*(b-a), 0, -d^2*a*b, 0])
[E.isogeny(E(te)).codomain() for te in E.torsion_subgroup()]
```

and then using SageMath's `E.conductor()`, `E.tamagawa_product()` to compute the conductor and the Tamagawa product (aborting the computation for each curve when it took more than 1 minute). This gave the following results, made visible in Figure 1 as blue dots.

- 841 curves have $q_\tau > 1.5$,
- 28 curves have $q_\tau > 2$,
- the curve with largest $q_\tau = 2.39875$ is
  $y^2 + xy = x^3 - 2713479277841926834110x - 5367476241939319246478821531590 0$,
  with $N = 105872910 = 2\,3\,5\,11\,13\,23\,29\,37$ and $\tau = 3981312 = 2^{14}\,3^5$,
  isogenous to the Frey-Hellegouarch curve, twisted by -1, for the *abc*-triple with
  $a = 22771715409 = 3^{16}\,23^2$,
  $b = 348972425216 = 2^{13}\,29^2\,37^3$,
  $c = 371744140625 = 5^9\,11^4\,13$,
  $q(a, b, c) = 1.44181$, $m(a, b, c) = 10.5196$,
- the curve with largest $\tau = 152202903552 = 2^{28}\,3^4\,7$ is
  $y^2 + xy = x^3 - 243293616838005191387643029131295594469691482466549330x-$
  $4618959831330247534541335903629393154870500982980307925945607057583704984500 7100$,
  with $q_\tau = 2.18988$ and $N = 251808730359756414 90 = 2\,3\,5\,11\,17\,19\,23\,37\,43\,61\,127\,173\,4817$,
  isogenous to the Frey-Hellegouarch curve, twisted by -1, for the *abc*-triple with
  $a = 4479069238054806835937 5 = 5^9\,17^2\,23^4\,37^2\,43\,4817$,
  $b = 34173001833284648695700365 29 = 3^{14}\,11^8\,61^2\,173^4$,
  $c = 34173449740208454176383959 04 = 2^{52}\,19^6\,127^2$,
  $q(a, b, c) = 1.41918$, $m(a, b, c) = 29.8237$,
  and also for the *abc*-triple with
  $a = 1467673944852224241 = 23^8\,37^4$
  $b = 1366929031440508578544641638 4 = 2^{28}\,3^7\,11^4\,19^3\,61\,127\,173^2$
  $c = 136692903145518531799316406 25 = 5^{18}\,17^4\,43^2\,4817^2$
  $q(a, b, c) = 1.45022$, $m(a, b, c) = 34.4028$, twisted by -1;
  this peculiar situation is explained in Section 3.6.
- Full tables `output_high_quality_qua.txt`, `output_high_quality_tam.txt` are on [dW3].

3.3 Curves from medium quality *abc*-triples

There are 1947 medium quality *abc*-triples known. They have been extracted from the two big tables `triples_below_1018_revised` (all 14482065 *abc*-triples below 10^{18}) and `big_triples` (an additional 9345651 *abc*-triples between 10^{18} and 2^{63}). With those medium quality *abc*-triples I did exactly the same as I did with the high quality *abc*-triples. This gave the following results, made visible in Figure 1 as red dots.

- 6342 curves have $q_\tau > 1.5$,
- 172 curves have $q_\tau > 2$,
- the curve with largest $q_\tau = 2.39177$ is
  $y^2 + xy = x^3 - 986143769212695065x - 3769280457563127484657527 75$,
  with $N = 13232310 = 2\,3\,5\,7\,13\,37\,131$ and $\tau = 1228800 = 2^{14}\,3\,5^2$,
  isogenous to the Frey-Hellegouarch curve, twisted by -1, for the *abc*-triple with
  $a = 658489 = 13\,37^3$,
  $b = 6879707136 = 2^{20}\,3^8$,
  $c = 6880365625 = 5^5\,7^5\,131$,
  $q(a, b, c) = 1.38137$, $m(a, b, c) = 6.67124$,
- the curves (isogenous) with largest $\tau = 509607936 = 2^{21}\,3^5$ are
  $y^2 + xy = x^3 - 13290632950903796089218578404113705-$

589748175639869043839018535079512741047463438442023, and
$y^2 + xy = x^3 - 13300785823649058521269530800913705-$
588802020491225519147911238670522467018762520682023,
both with $q_\tau = 2.19900$ and $N = 44947841915130 = 2\,3\,5\,7\,11\,17\,19\,23\,59\,103\,431$,
isogenous to the Frey-Hellegouarch curve, twisted by -1, for the *abc*-triple with
$a = 40675641638471 = 7^3\,17^9$,
$b = 798697622664921529 = 11^3\,19^3\,23\,59^2\,103^3$,
$c = 798738298306560000 = 2^{20}\,3^8\,5^4\,431^2$,
$q(a,b,c) = 1.31127$, $m(a,b,c) = 10.5021$.

- Full tables `output_medium_quality_qua.txt`, `output_medium_quality_tam.txt` are on [dW3].

3.4 Curves from high merit *abc*-triples

There are 202 high merit *abc*-triples available on [dS]. For almost all I succeeded to do the same procedure described above for high and medium quality *abc*-triples. This produced the following results, made visible in Figure 1 as yellow dots.

- 190 curves have $q_\tau > 1.5$,
- 172 curves have $q_\tau > 2$,
- the curve with largest $q_\tau = 2.18988$ is the same as found with the largest τ for the curves coming from high quality *abc*-triples,
- the curve with largest $\tau = 30644423884800 = 2^{25}\,3^4\,5^2\,11\,41$ is
  $y^2+xy+y = x^3-x^2-621574482712904069167623787332097562003319547609729892578\backslash$
  $282444666996972826570320570862x-5964690030130337213799773148403012416346828\backslash$
  $0237789284970994254311306494753746869776095054625199343391902064488625755369\backslash$
  77569403651,
  with $q_\tau = 1.70510$ and $N = 43081596887429422193675039055866970 =$
  $2\,3^2\,5\,7\,11\,17\,19\,29\,43\,73\,83\,97\,103\,151\,577\,751\,3167\,1230379$,
  isogenous to the Frey-Hellegouarch curve, twisted by -3, for the *abc*-triple with
  $a = 695606563606442148006101677581923 = 73^3\,97^2\,103^4\,577\,751\,3167\,1230379$,
  $b = 57576591665034362126590541368210176398589952 = 2^{45}\,17\,19^{10}\,29^4\,43^5\,151$,
  $c = 57576591665729968690196983516216278076171875 = 3^{11}\,5^{33}\,7^9\,11^2\,83^3$,
  $q(a,b,c) = 1.28114$, $m(a,b,c) = 27.1356$.
- Full tables `output_high_merit_qua.txt`, `output_high_merit_tam.txt` are on [dW3].

3.5 Curves from unbeaten *abc*-triples

Finally, there are 160 unbeaten *abc*-triples available on [dS], not necessarily different from curves in categories I have found above. Because those *abc*-triples quickly get amazingly large, I was only able to process the 30 ones with smallest value of c, and this did not yield any examples not already found above in other categories. Full tables `output_unbeaten_qua.txt`, `output_unbeaten_tam.txt` are on [dW3].

3.6 Curves from triples from triples

There is a nice trick to create new, hopefully better quality, *abc*-triples from triples of lesser quality, provided some miracle occurs. Assume a, b, c is an *abc*-triple of reasonable quality, and write $d = a + c$, $e = b + c$. Then look at the derived triples a, c, d and b, c, e, and hope for the miracle that one of them is of reasonable quality as well, maybe even of quality > 1 so that it is an *abc*-triple again. In that case I show how to get new triples of probably better quality than $q(a, b, c)$.

Note that $a < b < c < d < e$, and observe that

$$\left\{\begin{array}{l} c+a=d \\ c-a=b \end{array}\right. \iff \left\{\begin{array}{l} d+b=2c \\ d-b=2a \end{array}\right., \qquad \left\{\begin{array}{l} c+b=e \\ c-b=a \end{array}\right. \iff \left\{\begin{array}{l} e+a=2c \\ e-a=2b \end{array}\right. .$$

Now put

$$\begin{array}{rcll} (A_1, B_1, C_1) & = & (a^2, bd, c^2), & \\ (A_2, B_2, C_2) & = & (b^2, 4ac, d^2) & \text{(if } b \text{ is even, divide by 4; if } b^2 > 4ac\text{, swap)}, \\ (A_3, B_3, C_3) & = & (b^2, ae, c^2) & \text{(if } b^2 > ae\text{, swap)}, \\ (A_4, B_4, C_4) & = & (a^2, 4bc, e^2) & \text{(if } a \text{ is even, divide by 4)}. \end{array}$$

Clearly all four new triples have $A_i + B_i = C_i$ and $\gcd(A_i, B_i) = 1$ and $A_i < B_i$, and to find out if they are *abc*-triples only their quality has to be checked. Compared to the *abc*-triple a, b, c, the numerator of the quality function almost doubles, namely from at most $\log e < \log c + \log 2$ to at least $\log \frac{1}{4}c^2 = 2\log c - 2\log 2$. The denominator however also increases, but most probably by a factor smaller than 2, as it grows from a three-term radical like $r(a, b, c)$ to a four-term radical like $r(a, b, c, d)$. In an ideal case where $r(a) \approx r(b) \approx r(c) \approx r(d) \approx r(e)$ this growth factor in the denominator is $\approx 4/3$. So in such an ideal case the quality goes up by a factor $\approx \dfrac{2}{4/3} = \dfrac{3}{2}$. No practical case is ideal, but I will give some examples where the idea bears fruit, produces high-quality or high merit *abc*-triples, which in turn produce Frey-Hellegouarch curves with high Tamagawa quality.

Note that also $a + e = 2c$ and $b + d = 2c$, so that computing A_1, B_1, C_1 from $a, 2b, e$ gives the same result as computing A_4, B_4, C_4 from a, b, c, and computing A_2, B_2, C_2 from $2a, b, d$ gives the same result as computing A_3, B_3, C_3 from a, b, c.

This idea of creating triples from triples is not new, it occurs in J.P. van der Horst's master thesis [vdH].

I tried this out for all *abc*-triples found on [dS]. This resulted in 13 examples with quality above 1.4 or merit above 24. They are shown by black circles in Figure 1. Needless to say that they were all already present in these tables, so no new interesting curves with respect to Tamagawa quality were found, although the larger examples certainly lead to large τ and q_τ. But I found three cases of interest.

The first is from $a = 10 = 2\,5$, $b = 2187 = 3^7$, $c = 2197 = 13^3$, with $q(a, b, c) = 1.28975$, which for $e = 4384 = 2^5\,137$ leads to a not too bad quality $q(b, c, e) = 0.90396$, and then produces two high quality *abc*-triples:
$A_3 = 43840 = 2^6\,5\,137$, $B_3 = 4782969 = 3^{14}$, $C_3 = 4826809 = 13^6$,
with $q(A_3, B_3, C_3) = 1.41370$, and
$A_4 = 25 = 5^2$, $B_4 = 4804839 = 3^7\,13^3$, $C_4 = 4804864 = 2^8\,137^2$,
with $q(A_4, B_4, C_4) = 1.41328$.

The second is a similar case, from $a = 383102329 = 23^4\,37^2$, $b = 58457678566023 = 3^7\,11^4\,61\,173^2$, $c = 58458061668352 = 2^{26}\,19^3\,127$, with $q(a,b,c) = 1.13257$, which for $e = 116915740234375 = 5^9\,7^2\,43\,4817$ leads to a not too bad quality $q(b,c,e) = 0.854092$, and then produces two high quality and high merit *abc*-triples:
$A_3 = 44790692380548068359375 = 5^9\,17^2\,23^4\,37^2\,43\,4817$,
$B_3 = 3417300183328464869570036529 = 3^{14}\,11^8\,61^2\,173^4$,
$C_3 = 3417344974020845417638395904 = 2^{52}\,19^6\,127^2$,
with $q(A_3,B_3,C_3) = 1.41918$, $m(A_3,B_3,C_3) = 29.8237$, and
$A_4 = 146767394485224241 = 23^8\,37^4$,
$B_4 = 13669290314405085785446416384 = 2^{28}\,3^7\,11^4\,19^3\,61\,127\,173^2$,
$C_4 = 13669290314551853179931640625 = 5^{18}\,17^4\,43^2\,4817^2$,
with $q(A_4,B_4,C_4) = 1.45022$, $m(A_4,B_4,C_4) = 34.4028$.
Those two *abc*-triples appeared above, at the curve with the largest Tamagawa product found from high quality *abc*-triples as well as the largest Tamagawa quality found from high merit *abc*-triples.

The third is again a similar case, from $a = 158810997195450625 = 5^4\,53^2\,67^6$,
$b = 4025783917396764928 = 2^8\,23^2\,61^6\,577$, $c = 4184594914592215553 = 17^6\,311\,823^3$, with $q(a,b,c) = 1.08916$, which for $d = 4343405911787666178 = 2\,3^4\,7\,13\,109^4\,2087219$ leads to a not too bad quality $q(a,c,d) = 0.847863$, and then produces two high merit *abc*-triples:
$A_1 = 25220932830213426279247196812890625 = 5^8\,53^4\,67^{12}$,
$B_1 = 17485613666400818352222466948402205184 = 2^9\,3^4\,7\,13\,23^2\,61^6\,109^4\,577\,20872194$,
$C_1 = 17510834599231031778501714145215095809 = 17^{12}\,311^2\,823^6$,
with $m(A_1,B_1,C_1) = 30.0604$, and
$A_2 = 664559691245401291839706490968570625 = 5^4\,17^6\,53^2\,67^6\,311\,823^3$,
$B_2 = 4051734037392610655275387090022711296 = 2^{14}\,23^4\,61^{12}\,577^2$,
$C_2 = 4716293728638011947115093580991281921 = 3^8\,7^2\,13^2\,109^8\,20872194^2$,
with $m(A_2,B_2,C_2) = 26.5098$.

Full tables `output_triples_from_triples_qua.txt`, `output_triples_from_triples_tam.txt` are on [dW3].

4 Discussion

database	# curves	# *abc*-triples	# curves with $q_\tau > 1.5$	largest q_τ	largest τ
Cremona	3064705		10795	2.30681	87040
LMFDB	759667		0	1.22859	576
high quality		241	841	2.39875	152202903552
medium quality		1947	6342	2.39177	509607936
high merit		202	190	2.18988	30644423884800
unbeaten		160	24	2.18988	20615843020800
triples from triples		13	42	2.18988	1594506608640
all together			17760	2.39875	30644423884800

Table 1: Overview of found data on curves from different sources. Full data on [dW3].

For a summary of my results, see Table 1, and also Figure 1. Note that not all collections are necessarily disjunct.

As a first conclusion it seems justified to state that Frey-Hellegouarch curves for good *abc*-triples are a good source for high quality Tamagawa products. Finding large τ is probably not that hard, just by looking at curves with a large number of bad primes; the curve with largest τ found here ($\tau \approx 3 \times 10^{13}$) is a good example of that, having 18 bad primes. But finding curves with a high Tamagawa quality is another matter.

One might describe Pasten's conjecture 2 as cited above by the simple inequality

$$\limsup_{E/\mathbb{Q}} q_\tau \leq \frac{7}{3} \log 3.$$

Pasten already remarks that it could be open for improvement, and my experiments so far seem not to enthousiastically support a conjecture that $\frac{7}{3} \log 3$ is the true constant. As may be clear I am also interested in a lower bound for $\limsup_{E/\mathbb{Q}} q_\tau$. My experiments do not shed much light on this problem. It seems plausible to me to conjecture that at the very least $\limsup_{E/\mathbb{Q}} q_\tau > 0$, and I leave it to others to elaborate, and to come up with an argumented conjectured value, or at least a positive lower bound, for $\limsup_{E/\mathbb{Q}} q_\tau$. It does seem that Frey-Hellegouarch curves show split multiplicative reduction at a substantial portion of the bad primes, causing the local Tamagawa numbers at such primes being the exponents of the primes in the discriminant $16(abc)^2$ and thus contributing to large τ's, and this gives some hope that Pasten's analysis points in the right direction.

I also leave it for others to investigate whether the methods of papers searching for large Ш's, in particular [N], [DW] and [B], now geared towards small τ for obvious reasons, can be adapted to searching for big τ's at not too big conductors as well.

The code I wrote for this little project is so embarrassingly trivial that I did not want to set up an online repository for it. Some of the most essential code snippets are mentioned in the text above; everything else is simple data manipulation.

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